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Note

On the Size of the Coefficients of Rational Functions Approximating Powers, II

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We prove here the following:

THEOREM 1. *Let* $\frac{1}{2} \leqslant \alpha < 1$, $a_0, a_1, ..., a_n, b_0, b_1, ..., b_n$ $(n \geqslant 1)$ *be complex numbers where either* $a_0 = 0$ *or* $|a_0| \geqslant 1$ *and where* $|b_0| \geqslant 1$, *so that* $\sum_{k=0}^{n} b_k x^k \neq 0$ *throughout* $[0, 1]$ *and*

$$\max_{0 \leqslant x \leqslant 1} \left| x^\alpha - \frac{\sum_{k=0}^{n} a_k x^k}{\sum_{k=0}^{n} b_k x^k} \right| < \varepsilon, \qquad 0 < 2\varepsilon \leqslant 4^{-\alpha} \leqslant 2^{-1}.$$

Then,

$$\mu = \max(|a_1|, ..., |a_n|; |b_0|, ..., |b_n|) > \frac{4\varepsilon}{7(2\varepsilon)^{1/\alpha}}.$$

Remark. For $\alpha = \frac{1}{2}$, Theorem B of [1] follows.

Proof. If $a_0 \neq 0$, then $|a_0/b_0| < \varepsilon$, and hence

$$\mu \geqslant |b_0| > \frac{|a_0|}{\varepsilon} \geqslant \frac{1}{\varepsilon} > \frac{\varepsilon}{(2\varepsilon)^{1/\alpha}} > \frac{4\varepsilon}{7(2\varepsilon)^{1/\alpha}}.$$

Suppose $a_0 = 0$. Then

$$\left| \sum_{k=0}^{n} a_k (2\varepsilon)^{k/\alpha} \right| \leqslant \sum_{k=1}^{n} |a_k| (2\varepsilon)^{k/\alpha} < (2\varepsilon)^{1/\alpha} \mu \sum_{k=0}^{\infty} (2\varepsilon)^{k/\alpha} \leqslant \tfrac{4}{3} \mu (2\varepsilon)^{1/\alpha}.$$

$$\left| \sum_{k=0}^{n} b_k (2\varepsilon)^{k/\alpha} \right| \geqslant |b_0| - \sum_{k=1}^{n} |b_k| (2\varepsilon)^{k/\alpha} > 1 - \mu\, \tfrac{4}{3} (2\varepsilon)^{1/\alpha}.$$

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If $1-\mu\,\frac{4}{3}(2\varepsilon)^{1/\alpha}\leqslant 0$, then

$$\mu\geqslant\frac{3}{4(2\varepsilon)^{1/\alpha}}>\frac{4\varepsilon}{7(2\varepsilon)^{1/\alpha}}.$$

Otherwise, taking $x=(2\varepsilon)^{1/\alpha}$,

$$\varepsilon>\left|2\varepsilon-\frac{\sum_{k=0}^{n}a_k(2\varepsilon)^{k/\alpha}}{\sum_{k=0}^{n}b_k(2\varepsilon)^{k/\alpha}}\right|\geqslant 2\varepsilon-\frac{|\sum_{k=0}^{n}a_k(2\varepsilon)^{k/\alpha}|}{|\sum_{k=0}^{n}b_k(2\varepsilon)^{k/\alpha}|}>2\varepsilon-\frac{4\mu(2\varepsilon)^{1/\alpha}}{3-4\mu(2\varepsilon)^{1/\alpha}}$$

and hence

$$5\mu(2\varepsilon)^{1/\alpha}\geqslant(1+\varepsilon)\,4\mu(2\varepsilon)^{1/\alpha}>3\varepsilon,$$

$$\mu>\frac{3\varepsilon}{5(2\varepsilon)^{1/\alpha}}>\frac{4\varepsilon}{7(2\varepsilon)^{1/\alpha}}.$$

THEOREM 2. *Let $0<\alpha\leqslant 1$ and let $b_0, b_1, ..., b_n$ $(n\geqslant 1)$ be complex numbers such that $\sum_{k=0}^{n}b_kx^k\neq 0$ throughout $[0, 1]$ and*

$$\max_{0\leqslant x\leqslant 1}\left|x^\alpha-\left(\sum_{k=0}^{n}b_kx^k\right)^{-1}\right|<\varepsilon,\qquad 0<\varepsilon\leqslant\tfrac{1}{2}.$$

Then

$$\mu=\max(|b_0|, ..., |b_n|)>(4\varepsilon^\alpha)^{-1}.$$

Proof.

$$\varepsilon>\left|\left(\sum_{k=0}^{n}b_k\varepsilon^k\right)^{-1}-\varepsilon^\alpha\right|\geqslant\left|\sum_{k=0}^{n}b_k\varepsilon^k\right|^{-1}-\varepsilon^\alpha>(2\mu)^{-1}-\varepsilon^\alpha.$$

Hence

$$2\varepsilon^\alpha\geqslant\varepsilon^\alpha+\varepsilon>(2\mu)^{-1},$$

$$\mu>(4\varepsilon^\alpha)^{-1}.$$

REFERENCE

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